

On subexponential tails for the maxima of negatively driven compound renewal and Lévy processes

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Abstract

We study subexponential tail asymptotics for the distribution of the maximum $M_t := \sup_{u \in [0, t]} X_u$ of a process X_t with negative drift for the entire range of $t > 0$. We consider compound renewal processes with linear drift and Lévy processes. For both we also formulate and prove the principle of a single big jump for their maxima. The class of compound renewal processes with drift particularly includes Cramér–Lundberg renewal risk process.

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1 Introduction

For a probability distribution F on the real line, let $F(x) = F(-\infty, x]$ denote the distribution function and $\bar{F}(x) = F(x, \infty) = 1 - F(x)$ its tail. We say that F is (right-) heavy-tailed distribution if all its positive exponential moments are infinite, $\int_{\mathbb{R}} e^{sx} F(dx) = \infty$ for all $s > 0$. Otherwise we call F (right-) light-tailed.

In the presence of heavy tails, the class \mathcal{S} of subexponential distributions is of basic importance. A distribution F on \mathbb{R}^+ with unbounded support is called *subexponential* if $\overline{F * F}(x) \sim 2\bar{F}(x)$ as $x \rightarrow \infty$. A distribution F of ξ on the whole real line is called subexponential if the distribution F^+ of ξ^+ is so.

Any subexponential distribution is known (see, e.g., Foss et al. (2013, Lemma 3.2)) to be *long-tailed*, i.e., for any fixed y , $\bar{F}(x + y) \sim \bar{F}(x)$ as $x \rightarrow \infty$.

The class of subexponential distributions plays an important role in many applications, for instance, for waiting times in the $GI/G/1$ queue and for ruin probabilities—see, e.g., Asmussen (2003, Ch. X.9); Asmussen and Albrecher (2010, Ch. X); Embrechts et al. (1997, Sec. 1.4); Rolski et al. (1998).

A distribution F on \mathbb{R} with right unbounded support and finite mean is called *strong subexponential* ($F \in \mathcal{S}^*$) if

$$\int_0^x \bar{F}(x - y) \bar{F}(y) dy \sim 2\bar{F}(x) \int_0^\infty \bar{F}(y) dy \quad \text{as } x \rightarrow \infty.$$

It is known—see, e.g., Foss et al. (2013, Theorem 3.27)—that $F \in \mathcal{S}^*$ implies both $F \in \mathcal{S}$ and $F_I \in \mathcal{S}$ where F_I is the *integrated tail distribution* defined by its tail,

$$\bar{F}_I(x) := \min\left(1, \int_x^\infty \bar{F}(y) dy\right), \quad x > 0.$$

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Let Y, Y_1, Y_2, \dots be independent identically distributed random variables with a negative expectation $b = \mathbb{E}Y < 0$. Consider a random walk $S_0 = 0, S_n = Y_1 + \dots + Y_n$ and its maximum

$$M_n^S := \max_{0 \leq k \leq n} \sum_{i=1}^k Y_i,$$

hereinafter we follow the standard convention $\sum_{i=1}^0 f(i) = 0$.

Since $b < 0$, the family $M_n^S, n \geq 1$, is stochastically bounded. Let B be the distribution of Y_1^+ and B_I be the integrated tail distribution of Y_1^+ . As well known for the overall maximum of the random walk,

$$M_\infty^S = \max_{n \geq 0} \sum_{i=1}^n Y_i,$$

the asymptotic relation

$$\mathbb{P}\{M_\infty^S > x\} \sim \overline{B}_I(x)/|b| \quad \text{as } x \rightarrow \infty \quad (1)$$

holds in the heavy-tailed case if and only if the integrated tail distribution B_I is subexponential—see e.g. Theorem 5.12 in Foss et al. (2013). Also, if B is strong subexponential, $B \in \mathcal{S}^*$, then the following tail result holds for finite time horizon maxima

$$\mathbb{P}\left\{\max_{k \leq n} \sum_{i=1}^k Y_i > x\right\} \sim \frac{1}{|b|} \int_x^{x+n|b|} \overline{B}(v) dv \quad (2)$$

as $x \rightarrow \infty$ uniformly for all $n \geq 1$ —see Korshunov (2002) or Foss et al. (2013, Theorem 5.3); uniformity for all $n \geq 1$ means that

$$\sup_{n \geq 1} \left| \frac{\mathbb{P}\{\max_{k \leq n} \sum_{i=1}^k Y_i > x\}}{\frac{1}{|b|} \int_x^{x+n|b|} \overline{B}(v) dv} - 1 \right| \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

So the subexponential tail behaviour for the maxima of random walks is well understood while surprisingly much less is known for Lévy processes. In this contribution we particularly demonstrate in Section 2 how results for random walks relate to those for the compound renewal process with linear drift in the presence of heavy-tails—see Theorem 3; in particular, we formulate and prove the principle of a single big jump in Theorem 5. Based on that we give in Section 3 a very general treatment of subexponential tail behaviour for Lévy processes with negative drift—see Theorem 6. In Section 4 we derive tail asymptotics for a Lévy process stopped at random time and for its maximum within this time interval. An application to the Cramér–Lundberg renewal risk model is given in Section 5. A discussion of results available in the literature may be found just after Theorems 3 and 6.

2 Asymptotics for compound renewal process

Consider a *compound renewal process* X_t which is defined as

$$X_t = \sum_{i=1}^{N_t} Y_i,$$

where N_t is a renewal process generated by jump epochs $0 = T_0 < T_1 < T_2 < \dots$, where $\tau_n := T_n - T_{n-1} > 0$ are independent identically distributed random variables with finite mean $\mathbb{E}\tau =: 1/\lambda$, and Y_n , $n \geq 1$, are independent identically distributed jumps with finite mean b . The Y 's are supposed to be independent of the process N_t . Assume that the drift of the process is negative, that is, $b < 0$, so we have that the family of distributions of maxima

$$M_t := \max_{u \in [0, t]} X_u$$

is tight,

$$\sup_{t > 0} \mathbb{P}\{M_t > x\} \leq \mathbb{P}\{M_\infty > x\} \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

We are interested in the tail behaviour of M_t . The overall maximum M_∞ is simply the maximum of the associated random walk:

$$M_\infty = M_\infty^S = \max_{n \geq 0} \sum_{i=1}^n Y_i,$$

due to piecewise constant behaviour of the process X_t . Let B be the distribution of Y_1^+ and B_I be the integrated tail distribution of Y_1^+ . Then it follows from the result for the overall maximum of the associated random walk that

$$\mathbb{P}\{M_\infty > x\} \sim \overline{B}_I(x)/|b| \quad \text{as } x \rightarrow \infty \quad (3)$$

holds in the heavy-tailed case if and only if the integrated tail distribution B_I is subexponential.

The finite time horizon tail asymptotics for M_t are slightly more complicated than that for the infinite time horizon and are described in the following theorem.

Theorem 1. *Let X_t be a compound renewal process with negative drift $b/\mathbb{E}\tau < 0$. If the distribution B of Y_1^+ is strong subexponential, then, uniformly for all $t > 0$,*

$$\mathbb{P}\{M_t > x\} \sim \frac{1}{|b|} \int_x^{x+|b|\mathbb{E}N_t} \overline{B}(v) dv \quad \text{as } x \rightarrow \infty.$$

In particular,

$$\mathbb{P}\{M_t > x\} \sim \frac{1}{|b|} \int_x^{x+|b|\lambda t} \overline{B}(v) dv \quad \text{as } x, t \rightarrow \infty.$$

For a *compound Poisson process* X_t where N_t is a homogeneous Poisson process with intensity of jumps λ , we have $\mathbb{E}N_t = t\lambda$, so the following corollary.

Corollary 2. *Let X_t be a compound Poisson process with negative drift $\lambda b < 0$. If the distribution B of Y_1^+ is strong subexponential, then, uniformly for all $t > 0$,*

$$\mathbb{P}\{M_t > x\} \sim \frac{1}{|b|} \int_x^{x+t|b|\lambda} \overline{B}(v) dv \quad \text{as } x \rightarrow \infty.$$

Theorem 1 follows from a more general result stated next. It concerns a *compound renewal process with linear drift*, that is,

$$X_t = \sum_{i=1}^{N_t} Y_i + ct,$$

where N_t and the Y 's are as above while c is some real constant. Notice that the random variables $Y_i + c\tau_i$ depend on N_t . We assume that the drift of the process is negative, that is, $c + b\lambda < 0$ which implies that the family of distributions of maxima $M_t := \max_{u \in [0, t]} X_t$ is tight.

Theorem 3. *Let X_t be a compound renewal process with linear drift such that $a := c/\lambda + b < 0$. Let the distribution B of Y_1^+ be strong subexponential and one of the following conditions hold:*

- (i) $c \leq 0$;
- (ii) $c > 0$ and $\mathbb{P}\{c\tau > x\} = o(\overline{B}(x))$ as $x \rightarrow \infty$.

Then, uniformly for all $t > 0$,

$$\mathbb{P}\{M_t > x\} \sim \frac{1}{|a|} \int_x^{x+|a|\mathbb{E}N_t} \overline{B}(v) dv \quad \text{as } x \rightarrow \infty. \quad (4)$$

In particular,

$$\mathbb{P}\{M_t > x\} \sim \frac{1}{|a|} \int_x^{x+|a|\lambda t} \overline{B}(v) dv \quad \text{as } x, t \rightarrow \infty.$$

A particular case of this result was proven in Foss et al. (2013) by alternative techniques in the context of *Cramér–Lundberg collective risk model* where N_t is a Poisson process and $c < 0$ —see Theorem 5.21 there. In the book by Borovkovs (2008, Ch. 16) the tail behavior of M_t is only described for $t \rightarrow \infty$ and for regularly varying distribution of Y_1 .

If the linear drift coefficient is positive, that is $c > 0$, and if the condition $\mathbb{P}\{c\tau > x\} = o(\overline{B}(x))$ fails, then the tail asymptotics of the distribution of M_t may be more complicated. In particular, then $\mathbb{P}\{M_t > x\} \geq \mathbb{P}\{\tau_1 > x/c\}$, so the tail of M_t may be heavier than the integrated tail of B if the tail of τ is so. We do not concern tail asymptotics for M_t in the general case when $c > 0$; we only present the following result on the overall maximum M_∞ :

Theorem 4. *Let X_t be a compound renewal process such that $a := c/\lambda + b < 0$ and the integrated tail distribution F_I of $c\tau_1 + Y_1^+$ is subexponential. Then*

$$\mathbb{P}\{M_\infty > x\} \sim \frac{1}{|a|} \int_x^\infty \overline{F}(v) dv \quad \text{as } x \rightarrow \infty.$$

Notice that the distribution of $c\tau_1 + Y_1$ is strong subexponential in the case $c \leq 0$ if and only if the distribution of Y_1 is strong subexponential.

Proof of Theorem 3. First let us prove that, for any fixed t_0 , (4) holds uniformly for all $t \leq t_0$. Indeed, for all $t \leq t_0$,

$$\mathbb{P}\left\{\sum_{n=1}^{N_t} Y_n > x + |c|t_0\right\} \leq \mathbb{P}\{M_t > x\} \leq \mathbb{P}\left\{\sum_{n=1}^{N_t} Y_n^+ > x - |c|t_0\right\}.$$

Since the Y 's are strong subexponential, they are particularly subexponential. For the renewal process N_t , there exists a $\delta > 0$ such that

$$\sup_{t \leq t_0} \mathbb{E}(1 + \delta)^{N_{t_0}} < \infty.$$

Together with independence of the Y 's and N_t , it allows to apply Kesten's bound—see e.g. Foss et al. (2013, Theorem 3.34)—and to conclude the following uniform in $t \leq t_0$ analogue of the tail result for randomly stopped sums—see Foss et al. (2013, Theorem 3.37):

$$\mathbb{P}\left\{\sum_{n=1}^{N_t} Y_n > x\right\} \sim \mathbb{E}N_t \mathbb{P}\{Y_1 > x\} \quad \text{as } x \rightarrow \infty \text{ uniformly for all } t \leq t_0.$$

The same arguments work for the Y^+ 's. Therefore,

$$(1 + o(1))\mathbb{E}N_t \mathbb{P}\{Y_1 > x + |c|t_0\} \leq \mathbb{P}\{M_t > x\} \leq (1 + o(1))\mathbb{E}N_t \mathbb{P}\{Y_1 > x - |c|t_0\}$$

as $x \rightarrow \infty$ uniformly for all $t \leq t_0$. Subexponentiality of Y 's implies B is long-tailed, so hence

$$\mathbb{P}\{M_t > x\} \sim \mathbb{E}N_t \mathbb{P}\{Y_1 > x\} \quad \text{as } x \rightarrow \infty \text{ uniformly for all } t \leq t_0,$$

which is equivalent to the fact that (4) holds uniformly for all $t \leq t_0$ because

$$\frac{1}{|b|} \int_x^{x+|b|\mathbb{E}N_t} \overline{B}(v) dv \sim \mathbb{E}N_t \overline{B}(x) \quad \text{as } x \rightarrow \infty \text{ uniformly for all } t \leq t_0,$$

again by long-tailedness of B .

Therefore there exists an increasing function $h(x) \rightarrow \infty$ such that (4) holds uniformly for all $t \leq h(x)$.

Then it remains to prove (4) for the range $t > h(x)$ where the above arguments clearly do not help. Instead, we proceed with a standard technique of getting the lower and upper bounds for the tail of M_t which are asymptotically equivalent. For

the lower bound, fix an $\varepsilon > 0$. By the strong law of large numbers, there exists an A such that

$$\mathbb{P}\{|T_n - n\mathbb{E}\tau| < n\varepsilon + A \text{ for all } n \geq 1\} \geq 1 - \varepsilon. \quad (5)$$

Notice that

$$\mathbb{P}\{M_t > x\} \geq \mathbb{P}\left\{\sum_{i=1}^n Y_i + cT_n > x \text{ for some } n \leq N_t\right\}.$$

On the event (5), if $t \geq n(\mathbb{E}\tau + \varepsilon) + A$ (equivalently, $n \leq \lfloor \frac{t-A}{\mathbb{E}\tau + \varepsilon} \rfloor =: n(t)$) then $T_n \leq t$ and hence $n \leq N_t$. Since the jumps Y 's do not depend on the renewal process N_s , we obtain the inequality

$$\mathbb{P}\{M_t > x\} \geq (1 - \varepsilon)\mathbb{P}\left\{\sum_{i=1}^n Y_i + c(n(\mathbb{E}\tau + \varepsilon) + A) > x \text{ for some } n \leq n(t)\right\}$$

for $c \leq 0$ and the inequality

$$\mathbb{P}\{M_t > x\} \geq (1 - \varepsilon)\mathbb{P}\left\{\sum_{i=1}^n Y_i + c(n(\mathbb{E}\tau - \varepsilon) - A) > x \text{ for some } n \leq n(t)\right\}$$

for $c > 0$. Thus, in both cases,

$$\mathbb{P}\{M_t > x\} \geq (1 - \varepsilon)\mathbb{P}\left\{\max_{0 \leq n \leq n(t)} \sum_{i=1}^n (Y_i + c\mathbb{E}\tau - |c|\varepsilon) > x + |c|A\right\}.$$

Applying the equivalence (2) we obtain the following lower bound:

$$\begin{aligned} \mathbb{P}\{M_t > x\} &\geq \frac{1 - \varepsilon + o(1)}{|b + c\mathbb{E}\tau - |c|\varepsilon|} \int_{x+|c|A}^{x+|c|A+n(t)|b+c\mathbb{E}\tau-|c|\varepsilon|} \overline{B}(v) dv \\ &\sim \frac{1 - \varepsilon}{|a - |c|\varepsilon|} \int_0^{t \frac{|a-|c|\varepsilon|}{\mathbb{E}\tau + \varepsilon}} \overline{B}(x+v) dv \quad \text{as } x, t \rightarrow \infty, \end{aligned}$$

because B is a long-tailed distribution. Taking into account that, for every $\gamma > 0$,

$$\int_0^{\gamma t} \overline{B}(x+u) du \geq \min(1, \gamma) \int_0^t \overline{B}(x+u) du,$$

we conclude that

$$\mathbb{P}\{M_t > x\} \geq \frac{1 - \varepsilon + o(1)}{|a - |c|\varepsilon|} \min\left(1, \frac{|a - |c|\varepsilon| \mathbb{E}\tau}{\mathbb{E}\tau + \varepsilon} \frac{1}{|a|}\right) \int_x^{x+|a|t/\mathbb{E}\tau} \overline{B}(v) dv$$

as $x, t \rightarrow \infty$. Letting $\varepsilon \downarrow 0$ completes the proof of the lower bound

$$\mathbb{P}\{M_t > x\} \geq \frac{1 + o(1)}{|a|} \int_x^{x+|a|t/\mathbb{E}\tau} \overline{B}(v) dv \quad \text{as } x, t \rightarrow \infty.$$

Now let us turn to the upper bound for $\mathbb{P}\{M_t > x\}$. First consider the case $c \leq 0$ when the trajectory of X_t linearly drops down between jumps and the maximum may be only attained at a jump epoch,

$$M_t = \max_{0 \leq n \leq N_t} \sum_{i=1}^n (Y_i + c\tau_i).$$

Therefore, for any $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}\{M_t > x\} &\leq \mathbb{P}\left\{\max_{0 \leq n \leq (1+\varepsilon)\mathbb{E}N_t} \sum_{i=1}^n (Y_i + c\tau_i) > x\right\} \\ &\quad + \mathbb{P}\left\{\sum_{i=1}^{N_t} Y_i^+ > x, N_t > (1+\varepsilon)\mathbb{E}N_t\right\}. \end{aligned} \quad (6)$$

The distribution of Y is strong subexponential and $c < 0$, so $Y + c\tau$ is strong subexponential too and

$$\mathbb{P}\{Y + c\tau > x\} \sim \mathbb{P}\{Y > x\} = \overline{B}(x) \quad \text{as } x \rightarrow \infty.$$

Thus, by (2),

$$\begin{aligned} \mathbb{P}\left\{\max_{0 \leq n \leq (1+\varepsilon)\mathbb{E}N_t} \sum_{i=1}^n (Y_i + c\tau_i) > x\right\} &\sim \frac{1}{|a|} \int_x^{x+|a|(1+\varepsilon)\mathbb{E}N_t} \overline{B}(y) dy \\ &\leq \frac{1+\varepsilon}{|a|} \int_x^{x+|a|\mathbb{E}N_t} \overline{B}(y) dy, \end{aligned} \quad (7)$$

because $\overline{B}(y)$ is decreasing. Further,

$$\begin{aligned} &\mathbb{P}\left\{\sum_{i=1}^{N_t} Y_i^+ > x, N_t > (1+\varepsilon)\mathbb{E}N_t\right\} \\ &= \sum_{k=1}^{\infty} \mathbb{P}\left\{\sum_{i=1}^{N_t} Y_i^+ > x, (1+k\varepsilon)\mathbb{E}N_t < N_t \leq (1+(k+1)\varepsilon)\mathbb{E}N_t\right\} \\ &\leq \sum_{k=1}^{\infty} \mathbb{P}\left\{\sum_{i=1}^{(1+(k+1)\varepsilon)\mathbb{E}N_t} Y_i^+ > x\right\} \mathbb{P}\{N_t > (1+k\varepsilon)\mathbb{E}N_t\}, \end{aligned} \quad (8)$$

owing to independence of Y 's and N_t . Denote $K := [(1+k\varepsilon)\mathbb{E}N_t]$. Then

$$\begin{aligned} \mathbb{P}\{N_t > (1+k\varepsilon)\mathbb{E}N_t\} &= \mathbb{P}\{T_K \leq t\} \\ &= \mathbb{P}\{K\mathbb{E}\tau(1-\varepsilon/2) - T_K \geq K\mathbb{E}\tau(1-\varepsilon/2) - t\} \\ &\leq \mathbb{P}\{K\mathbb{E}\tau(1-\varepsilon/2) - T_K \geq 0\} \end{aligned}$$

for sufficiently large t and $\varepsilon \in (0, 1)$ because, as $t \rightarrow \infty$,

$$\begin{aligned} K\mathbb{E}\tau(1-\varepsilon/2) - t &\sim t((1+k\varepsilon)(1-\varepsilon/2) - 1) \\ &\geq t(\varepsilon/2 - \varepsilon^2/2) > 0. \end{aligned}$$

Since the random variable $\mathbb{E}\tau(1 - \varepsilon/2) - \tau$ has negative expectation $-\varepsilon\mathbb{E}\tau/2$ and is bounded from above by $\mathbb{E}\tau(1 - \varepsilon/2)$, there exists a $\beta = \beta(\varepsilon) > 0$ such that

$$\mathbb{E}e^{\beta(\mathbb{E}\tau(1-\varepsilon/2)-\tau)} = 1 - \delta < 1.$$

Hence, by exponential Chebyshev's inequality,

$$\mathbb{P}\{K\mathbb{E}\tau(1 - \varepsilon/2) - T_K \geq 0\} \leq (1 - \delta)^K$$

for all $k \geq 1$ and sufficiently large t , so

$$\mathbb{P}\{N_t > (1 + k\varepsilon)\mathbb{E}N_t\} \leq (1 - \delta)^{[(1+k\varepsilon)\mathbb{E}N_t]}. \quad (9)$$

By Kesten's bound—see e.g. Foss et al. (2013, Theorem 3.34)—there is an $A < \infty$ such that

$$\mathbb{P}\left\{\sum_{i=1}^{(1+(k+1)\varepsilon)\mathbb{E}N_t} Y_i^+ > x\right\} \leq A(1 + \delta/8)^{(1+(k+1)\varepsilon)\mathbb{E}N_t} \mathbb{P}\{Y > x\}$$

for all $x > 0$, $k \geq 1$ and $t > 0$. For $k \geq 1$ and sufficiently large t ,

$$(1 + (k + 1)\varepsilon)\mathbb{E}N_t \leq 2[(1 + k\varepsilon)\mathbb{E}N_t],$$

thus

$$\begin{aligned} \mathbb{P}\left\{\sum_{i=1}^{(1+(k+1)\varepsilon)\mathbb{E}N_t} Y_i^+ > x\right\} &\leq A(1 + \delta/8)^{2[(1+k\varepsilon)\mathbb{E}N_t]} \mathbb{P}\{Y > x\} \\ &\leq A(1 + \delta/2)^{[(1+k\varepsilon)\mathbb{E}N_t]} \mathbb{P}\{Y > x\}. \end{aligned} \quad (10)$$

Substituting (9) and (10) into (8) and taking into account that $(1 - \delta)(1 + \delta/2) \leq 1 - \delta/2$, we obtain, for all sufficiently large t ,

$$\mathbb{P}\left\{\sum_{i=1}^{N_t} Y_i^+ > x, N_t > (1 + \varepsilon)\mathbb{E}N_t\right\} \leq A\mathbb{P}\{Y > x\} \sum_{k=1}^{\infty} (1 - \delta/2)^{[(1+k\varepsilon)\mathbb{E}N_t]}.$$

The sum on the right goes to zero as $t \rightarrow \infty$. Therefore, for any fixed $\varepsilon > 0$,

$$\mathbb{P}\left\{\sum_{i=1}^{N_t} Y_i^+ > x, N_t > (1 + \varepsilon)\mathbb{E}N_t\right\} = o(\mathbb{P}\{Y > x\})$$

as $t \rightarrow \infty$ uniformly for all $x > 0$. Combining this bound with (7) we get

$$\mathbb{P}\{M_t > x\} \leq \frac{1 + \varepsilon + o(1)}{|a|} \int_x^{x+|a|\mathbb{E}N_t} \overline{B}(y) dy$$

as $x \rightarrow \infty$ uniformly for $t \geq h(x)$. Letting $\varepsilon \downarrow 0$, we conclude

$$\mathbb{P}\{M_t > x\} \leq \frac{1 + o(1)}{|a|} \int_x^{x+|a|\mathbb{E}N_t} \overline{B}(v) dv$$

as $x \rightarrow \infty$ uniformly for $t \geq h(x)$. This proves Theorem 3 in the case $c \leq 0$.

Now consider the case $c > 0$ when the trajectory of X_t linearly grows between jumps and the maximum may be only attained just prior to a jump epoch or at time t , so hence

$$\begin{aligned} M_t &\leq c\tau_1 + \max_{0 \leq n \leq N_t} \left(\sum_{i=1}^n Y_i + c((T_{n+1} - \tau_1) \wedge t) \right) \\ &=: c\tau_1 + \widehat{M}_t, \end{aligned} \quad (11)$$

where τ_1 and \widehat{M}_t are independent. Similar to the case $c \leq 0$, for any $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}\{\widehat{M}_t > x\} &\leq \mathbb{P}\left\{ \max_{0 \leq n \leq (1+\varepsilon)\mathbb{E}N_t} \sum_{i=1}^n Y_i + ct > x \right\} \\ &\quad + \mathbb{P}\left\{ \sum_{i=1}^{N_t} Y_i^+ + ct > x, N_t > (1+\varepsilon)\mathbb{E}N_t \right\}. \end{aligned} \quad (12)$$

The distribution of Y is strong subexponential and the tail of $c\tau$ is of order $o(\overline{B}(x))$, so $Y + c\tau$ is strong subexponential too and

$$\mathbb{P}\{Y + c\tau > x\} \sim \mathbb{P}\{Y > x\} = \overline{B}(x) \quad \text{as } x \rightarrow \infty.$$

Thus, by (2), we get that the first term on the right hand side of (12) possesses the upper bound (7). The second term on the right hand side of (12) may be bounded from above as follows. Take c_1 so large that $c_1\mathbb{E}N_t \geq t$ for all $t > 1$. Then

$$\begin{aligned} &\mathbb{P}\left\{ \sum_{i=1}^{N_t} Y_i^+ + ct > x, N_t > (1+\varepsilon)\mathbb{E}N_t \right\} \\ &= \mathbb{P}\left\{ \sum_{i=1}^{N_t} (Y_i^+ + c_1) + ct - c_1N_t > x, N_t > (1+\varepsilon)\mathbb{E}N_t \right\} \\ &\leq \mathbb{P}\left\{ \sum_{i=1}^{N_t} (Y_i^+ + c_1) > x, N_t > (1+\varepsilon)\mathbb{E}N_t \right\}, \end{aligned}$$

which possesses the same upper bound as the second term on the right hand side of (6). Altogether it implies that

$$\mathbb{P}\{\widehat{M}_t > x\} \leq \frac{1 + o(1)}{|a|} \int_x^{x+|a|\mathbb{E}N_t} \overline{B}(y) dy \quad \text{as } x \rightarrow \infty.$$

Since $c\tau_1$ and \widehat{M}_t in (11) are independent,

$$\mathbb{P}\{M_t > x\} \leq \mathbb{P}\{c\tau_1 > x\} + \int_0^x \mathbb{P}\{c\tau_1 \in du\} \mathbb{P}\{\widehat{M}_t > x - u\}$$

which allows to carry out standard calculations for subexponential distributions based on the condition $\mathbb{P}\{c\tau_1 > x\} = o(\overline{B}(x))$ and the upper bound for \widehat{M}_t and to conclude the upper bound

$$\mathbb{P}\{M_t > x\} \leq \frac{1 + o(1)}{|a|} \int_x^{x+|a|\mathbb{E}N_t} \overline{B}(y) dy \quad \text{as } x \rightarrow \infty.$$

which completes the proof in the case $c > 0$. The proof of Theorem 3 is complete. \square

Proof of Theorem 4. We need only to consider the case $c > 0$. Then the lower bound for the tail of M_∞ follows from the inequality

$$M_\infty \geq \sup_{k \geq 0} \sum_{i=1}^k (Y_i + c\tau_{i+1}) =: \zeta$$

and from the result (3) for maxima of sums. The upper bound follows from the equality

$$M_\infty = c\tau_1 + \zeta$$

and from the observation that

$$\mathbb{P}\{c\tau_1 > x\} = O(\overline{F}(x)) = o(\overline{F}_I(x)) \quad \text{as } x \rightarrow \infty$$

which allows to apply [14, Corollary 3.18]. The proof is complete. \square

We conclude this section with the following theorem which is nothing other than the *principle of a single big jump* for the maximum M_t . For any $A > 0$ and $\varepsilon > 0$ consider events

$$D_k := \{|X_s - a\lambda s| \leq \varepsilon s + A \text{ for all } s < T_k, Y_k > x + |a|\lambda T_k\} \quad (13)$$

which, for large x , roughly speaking means that up to time T_k the process X_s drifts down with rate a according to the strong law of large numbers and then makes a big jump up at time T_k of size x plus value that compensates the negative drift up to this time. As stated in the next theorem, the union of these events describes the most probable way by which large deviations of M_t can occur—it is very different from what is observed if Y 's possess some positive exponential moment finite. It is an analogue for discrete time process of the principle of a single big jump for the maximum of a random walk with negative drift, see Theorem 5.4 in Foss et al. (2013).

Theorem 5. *In conditions of Theorem 3, for any fixed $\varepsilon > 0$,*

$$\lim_{A \rightarrow \infty} \lim_{t, x \rightarrow \infty} \mathbb{P}\{\cup_{k=1}^{N_t} D_k | M_t > x\} \geq \frac{|a|}{|a| + 2\varepsilon/\lambda}.$$

Proof. Choose $\gamma > 0$ so small that $(|a|\lambda + \varepsilon)(1/\lambda + \gamma\varepsilon) < |a| + 2\varepsilon/\lambda$ for all $\varepsilon \in (0, 1)$. Then, since, for k such that $k(\mathbb{E}\tau + \gamma\varepsilon) + A \leq t$, each of the events

$$\begin{aligned} \tilde{D}_k &:= \{ |X_s - a\lambda s| \leq \varepsilon s + A \text{ for all } s < T_k, \ T_j \leq j(\mathbb{E}\tau + \varepsilon\gamma) + A \text{ for all } j \leq k, \\ &\quad M_{T_k-0} \leq x, \ Y_k > x + A + T_k(|a|\lambda + \varepsilon) \} \end{aligned}$$

is contained in $T_k \leq t$ and in D_k and implies that $M_{T_k} > x$ because on the event \tilde{D}_k we have

$$\begin{aligned} X_{T_k} &= X_{T_k-0} + Y_k \\ &> (a\lambda - \varepsilon)T_k - A + x + A + T_k(|a|\lambda + \varepsilon) = x, \end{aligned}$$

so that $M_t > x$. Then, for $N := \lfloor \frac{t-A}{\mathbb{E}\tau + \gamma\varepsilon} \rfloor$, we consequently have that

$$\mathbb{P}\{\cup_{k=1}^{N_t} D_k | M_t > x\} \geq \mathbb{P}\{\cup_{k=1}^N \tilde{D}_k | M_t > x\} = \frac{\mathbb{P}\{\cup_{k=1}^N \tilde{D}_k\}}{\mathbb{P}\{M_t > x\}}. \quad (14)$$

The events \tilde{D}_k are disjoint, hence

$$\mathbb{P}\{\cup_{k=1}^N \tilde{D}_k\} = \sum_{k=1}^N \mathbb{P}\{\tilde{D}_k\}.$$

It follows from the strong law of large numbers applied to both X_s and N_s that, for any fixed $\delta > 0$, there exists an A such that, for all $x > A$,

$$\begin{aligned} \mathbb{P}\{\cup_{k=1}^N \tilde{D}_k\} &\geq (1 - \delta/4) \sum_{k=1}^N \mathbb{P}\{Y_k > x + A + T_k(|a|\lambda + \varepsilon) \mid T_k \leq k(\mathbb{E}\tau + \varepsilon\gamma) + A\} \\ &\geq (1 - \delta/4) \sum_{k=1}^N \mathbb{P}\{Y_k > x + (1 + |a|\lambda + \varepsilon)A + k(|a|\lambda + \varepsilon)(\mathbb{E}\tau + \varepsilon\gamma)\} \\ &\geq (1 - \delta/4) \sum_{k=1}^N \mathbb{P}\{Y_k > x + (1 + |a|\lambda + \varepsilon)A + k(|a| + 2\varepsilon/\lambda)\}, \end{aligned}$$

by the choice of the $\gamma > 0$. Since the distribution B is long-tailed,

$$\mathbb{P}\{\cup_{k=1}^N \tilde{D}_k\} \geq (1 - \delta/2) \sum_{k=0}^{N-1} \mathbb{P}\{Y_k > x + k(|a| + 2\varepsilon/\lambda)\}$$

for all sufficiently large x . Hence

$$\mathbb{P}\{\cup_{k=0}^{N-1} \tilde{D}_k\} \geq \frac{1 - \delta/2}{|a| + 2\varepsilon/\lambda} \int_x^{x+N(|a|+2\varepsilon/\lambda)} \overline{B}(y) dy,$$

because $\overline{B}(y)$ decreases. Take also into account that, for some $c_1 < \infty$,

$$N(|a| + 2\varepsilon/\lambda) \geq t \frac{|a| + 2\varepsilon/\lambda}{\mathbb{E}\tau + \gamma\varepsilon} - c_1 \geq t(|a|\lambda + \varepsilon) - c_1,$$

owing the choice of $\gamma > 0$, so

$$N(|a| + 2\varepsilon/\lambda) \geq t|a|\lambda \quad \text{for all sufficiently large } t.$$

Then we deduce

$$\mathbb{P}\{\cup_{k=1}^N \tilde{D}_k\} \geq \frac{1 - \delta/2}{|a| + 2\varepsilon/\lambda} \int_x^{x+t|a|\lambda} \overline{B}(y) dy.$$

Substituting this estimate and the asymptotics for M_t into (14) we deduce that

$$\lim_{t, x \rightarrow \infty} \mathbb{P}\{\cup_{k=1}^{N_t} D_k | M_t > x\} \geq \frac{(1 - \delta)|a|}{|a| + 2\varepsilon/\lambda}.$$

Now we can make $\delta > 0$ as small as we please by choosing a sufficiently large A . This completes the proof. \square

3 Asymptotics for Lévy process

Let X_t be a càdlàg stochastic process in \mathbb{R} which means that its paths are right continuous with left limits everywhere, with probability 1. Then, for every t , the supremum

$$M_t := \sup_{u \in [0, t]} X_u$$

is finite a.s. In this section we study tail behaviour of the distribution of M_t for a *Lévy process* X_t starting at the origin, that is, for a stochastic process with stationary independent increments, where stationary means that, for $s < t$, the probability distribution of $X_t - X_s$ depends only on $t - s$ and where independent increments means that that difference $X_t - X_s$ is independent of the corresponding difference on any interval not overlapping with $[s, t]$, and similarly for any finite number of mutually non-overlapping intervals. Our main result for Lévy processes is the following theorem.

Theorem 6. *Assume the finite mean and negative drift, $a := \mathbb{E}X_1 < 0$. If the integrated tail distribution F_I of X_1 is subexponential, then*

$$\mathbb{P}\{\max_{u > 0} X_u > x\} \sim \frac{1}{|a|} \int_x^\infty \overline{F}(v) dv \quad \text{as } x \rightarrow \infty.$$

If the distribution F of X_1 is strong subexponential, then, uniformly for all $t > 0$,

$$\mathbb{P}\{\max_{u \in [0, t]} X_u > x\} \sim \frac{1}{|a|} \int_x^{x+t|a|} \overline{F}(v) dv \quad \text{as } x \rightarrow \infty.$$

It has been suggested by Asmussen and Klüppelberg (1996) and by Asmussen (1998) to follow a discrete skeleton argument in order to prove this asymptotics for $t = \infty$ when the tail of the Lévy measure is subexponential; notice that this

approach requires additional considerations which take into account fluctuations of Lévy processes within time slots; see the remark after Theorem 7.

In Braverman et al. (2002) tail asymptotics are presented for some subclass of subadditive functionals of Lévy process with regularly varying at infinity Lévy measure. The overall supremum is a particular example considered in that article.

In Klüppelberg et al. (2004, Theorem 6.2), tail asymptotics for the overall supremum of negatively driven Lévy process are derived via direct approach based on ladder properties of the Lévy process.

In Doney et al. (2016) the passage time problem is considered for Lévy processes, emphasising heavy tailed cases; local and functional versions of limit distributions are derived for the passage time itself, as well as for the position of the process just prior to passage, and the overshoot of a high level which is an extension for Lévy processes of corresponding results for random walks, see e.g. Foss et al. (2013, Theorem 5.24).

In Foss et al. (2007, Theorem 3.1), Markov modulated Lévy process is studied and again the tail asymptotics for the overall supremum were proven, via reduction to Markov modulated random walk.

In the book by Borovkovs (2008, Ch. 15) some partial results on $\max_{u \in [0, t]} X_u$ are formulated (see, for example, Theorems 15.2.2(vi) and 15.3.12 there) under some specific conditions on the distribution of X_1 ; the supporting arguments provided may be hardly considered as clear and comprehensive. For example, on page 525 the authors justify transition from integer t to non-integer t by convergence in probability $X_u \rightarrow 0$ as $u \rightarrow \infty$ which is clearly insufficient. Also notice that it was not proven there that the corresponding asymptotics hold uniformly for all $t > 0$.

Related results on sample-path large deviations of scaled Lévy processes $X(nt)/n$ with regularly varying Lévy measure are proven by Rhee et al. (2016).

The following result is due to Willekens [23]; it was proven via natural elementary rather short arguments.

Theorem 7. *Let X_t be a Lévy process. For any fixed $t > 0$, the following assertions are equivalent:*

- (i) *the distribution of X_t is long-tailed;*
- (ii) *the distribution of M_t is long-tailed.*

Each of (i) and (ii) implies

$$\mathbb{P}\{M_t > x\} \sim \mathbb{P}\{X_t > x\} \quad \text{as } x \rightarrow \infty. \quad (15)$$

Notice that Theorem 7 together with Theorem 1 for regenerative processes from Palmowski and Zwart (2007)—or with Theorem 3.3 from Asmussen et al. (1999)—provides a correct version of skeleton approach for proving subexponential asymptotics for the overall supremum M_∞ under negative drift assumption.

In our proof of Theorem 6 we need the following lemma which may be of independent interest.

Lemma 8. *Let G and B be two distributions on \mathbb{R} and let G be light-tailed, that is, there exist $\lambda > 0$ and $c < \infty$ such that $\overline{G}(x) \leq ce^{-\lambda x}$ for all x . Denote $F := G * B$.*

(i) If B is long-tailed then $\overline{F}(x) \sim \overline{B}(x)$ as $x \rightarrow \infty$; in particular, F is long-tailed too.

(ii) If F is long-tailed then B is long-tailed too.

Similar proposition was proven for subexponential distributions in Embrechts et al. (1979, Proposition 1); our proof is similar.

Proof. (i) Assume that B is long-tailed. Then there exists an increasing function $h(x) \rightarrow \infty$ such that (see Foss et al. (2011, Lemma 2.19))

$$\overline{B}(x - h(x)) \sim \overline{B}(x) \quad \text{as } x \rightarrow \infty. \quad (16)$$

Consider the following decomposition:

$$\begin{aligned} \frac{\overline{F}(x)}{\overline{B}(x)} &= \int_{-\infty}^{h(x)} \frac{\overline{B}(x-y)}{\overline{B}(x)} G(dy) + \int_{h(x)}^{\infty} \frac{\overline{B}(x-y)}{\overline{B}(x)} G(dy) \\ &=: I_1(x) + I_2(x). \end{aligned}$$

Since

$$\frac{\overline{B}(x-y)}{\overline{B}(x)} \leq \frac{\overline{B}(x-h(x))}{\overline{B}(x)}$$

for all $y \leq h(x)$, it follows from (16) that the integrand in $I_1(x)$ possesses an integrable majorant. Moreover, for every y , $\frac{\overline{B}(x-y)}{\overline{B}(x)} \rightarrow 1$ as $x \rightarrow \infty$. Hence, by the dominated convergence theorem,

$$I_1(x) \rightarrow 1 \quad \text{as } x \rightarrow \infty. \quad (17)$$

Further, since the distribution B is long-tailed, for any $\varepsilon > 0$ there exists $x(\varepsilon)$ such that

$$\overline{B}(x-1) \leq \overline{B}(x)e^\varepsilon \quad \text{for all } x \geq x(\varepsilon).$$

Hence, there exists $c(\varepsilon) < \infty$ such that

$$\overline{B}(x-y) \leq c(\varepsilon)\overline{B}(x)e^{\varepsilon y} \quad \text{for all } x \geq x(\varepsilon), y > 0.$$

Take $\varepsilon < \lambda$. Then

$$I_2(x) \leq c(\varepsilon) \int_{h(x)}^{\infty} e^{\varepsilon y} G(dy) \rightarrow 0 \quad \text{as } x \rightarrow \infty,$$

because $\overline{G}(x) = O(e^{-\lambda x})$ and $h(x) \rightarrow \infty$ as $x \rightarrow \infty$. Together with (17) it implies the relation $\overline{F}(x) \sim \overline{B}(x)$ as $x \rightarrow \infty$.

(ii) Assume that F is long-tailed. Let us then prove that $\overline{B}(x) \sim \overline{F}(x)$ which implies long-tailedness of B . Since F is long-tailed, there exists a function $h(x) \rightarrow \infty$ such that $\overline{F}(x-h(x)) \sim \overline{F}(x)$ as $x \rightarrow \infty$.

For every x and $h \in \mathbb{R}$ the following inequality holds:

$$\overline{F}(x - h) = \overline{G * B}(x - h) \geq \overline{G}(-h)\overline{B}(x).$$

If we choose h_0 satisfying $\overline{G}(-h_0) \geq 1/2$ then

$$\overline{B}(x) \leq 2\overline{F}(x - h_0) \quad \text{for all } x \in \mathbb{R}. \quad (18)$$

Also we deduce that

$$\overline{B}(x) \leq \frac{\overline{F}(x - h(x))}{\overline{G}(-h(x))} \sim \overline{F}(x) \quad \text{as } x \rightarrow \infty.$$

So, it remains to prove that

$$\liminf_{x \rightarrow \infty} \frac{\overline{B}(x)}{\overline{F}(x)} \geq 1. \quad (19)$$

Suppose it does not hold. Then there exist an $\varepsilon > 0$ and a sequence $x_n \rightarrow \infty$ such that

$$\overline{B}(x_n - h_0) \leq (1 - \varepsilon)\overline{F}(x_n - h_0) \quad \text{for all } n \geq 1. \quad (20)$$

We have

$$\begin{aligned} \overline{F}(x_n) &= \int_{-\infty}^{h_0} \overline{B}(x_n - y)G(dy) + \int_{h_0}^{\infty} \overline{B}(x_n - y)G(dy) \\ &\leq \overline{B}(x_n - h_0) + \int_{h_0}^{\infty} \overline{B}(x_n - y)G(dy) \\ &\leq (1 - \varepsilon)\overline{F}(x_n - h_0) + 2 \int_{h_0}^{\infty} \overline{F}(x_n - h_0 - y)G(dy), \end{aligned}$$

by (20) and (18). Since the distribution F is assumed to be long-tailed, the calculations of part (i) show that

$$\int_{h_0}^{\infty} \overline{F}(x_n - h_0 - y)G(dy) \sim \overline{F}(x_n - h_0)\overline{G}(h_0) \quad \text{as } n \rightarrow \infty.$$

Therefore, for every h_0 satisfying $\overline{G}(-h_0) \geq 1/2$,

$$1 = \lim_{n \rightarrow \infty} \frac{\overline{F}(x_n)}{\overline{F}(x_n - h_0)} \leq 1 - \varepsilon + 2\overline{G}(h_0).$$

Letting $h_0 \rightarrow \infty$ leads to the contradiction $1 \leq 1 - \varepsilon$. This justifies (19) and the proof is complete. \square

Given X_1 has infinitely divisible distribution, recall the Lévy–Khintchine formula for the characteristic exponent $\Psi(\theta) := \log \mathbb{E}e^{i\theta X_1}$, for every $\theta \in \mathbb{R}$,

$$\begin{aligned}\Psi(\theta) &= \left(i\alpha\theta - \frac{1}{2}\sigma^2\theta^2\right) + \int_{0 < |x| < 1} (e^{i\theta x} - 1 - i\theta x)\Pi(dx) + \int_{|x| \geq 1} (e^{i\theta x} - 1)\Pi(dx) \\ &=: \Psi_1(\theta) + \Psi_2(\theta) + \Psi_3(\theta);\end{aligned}$$

see, e.g. Kyprianou (2006, Sect. 2.1). Here Π is the Lévy measure concentrated on $\mathbb{R} \setminus \{0\}$ and satisfying $\int_{\mathbb{R}} (1 \wedge x^2)\Pi(dx) < \infty$. Let $X_t^{(1)}$, $X_t^{(2)}$ and $X_t^{(3)}$ be independent processes given in the Lévy–Itô decomposition $X_t \stackrel{d}{=} X_t^{(1)} + X_t^{(2)} + X_t^{(3)}$, where $X_t^{(1)}$ is a linear Brownian motion with characteristic exponent given by $\Psi^{(1)}$, $X_t^{(2)}$ is a square integrable martingale with an almost surely countable number of jumps on each finite time interval which are of magnitude less than unity and with characteristic exponent given by $\Psi^{(2)}$ and $X_t^{(3)}$ is a compound Poisson process with intensity $\Pi(\mathbb{R} \setminus (-1, 1))$ and jump distribution $\frac{\Pi(dx)}{\Pi(\mathbb{R} \setminus (-1, 1))}$ concentrated on $(-\infty, -1) \cup (1, \infty)$. It is known—see, e.g. Kyprianou (2006, Theorem 3.6) or Sato (1999, Theorem 25.17)—that the sum $Z_t := X_t^{(1)} + X_t^{(2)}$ possesses all exponential moments finite,

$$\mathbb{E}e^{sZ_t} = \mathbb{E}e^{s(X_t^{(1)} + X_t^{(2)})} < \infty \quad \text{for all } s \in \mathbb{R}. \quad (21)$$

In particular, exponential moments of $X_t^{(2)}$ may be bounded as follows. By the condition $\int_{(-1, 1)} x^2 \Pi(dx) < \infty$ we may produce the following upper bound:

$$\begin{aligned}\int_{(-1, 1)} (e^{sx} - 1 - sx)\Pi(dx) &= \int_{(-1, 1)} \sum_{k=2}^{\infty} \frac{(sx)^k}{k!} \Pi(dx) \\ &\leq \sum_{k=2}^{\infty} \frac{s^k}{k!} \int_{(-1, 1)} x^2 \Pi(dx) \\ &= c(e^s - 1 - s),\end{aligned}$$

where $c := \int_{(-1, 1)} x^2 \Pi(dx)$. Therefore,

$$\mathbb{E}e^{sX_t^{(2)}} = e^{t \int_{(-1, 1)} (e^{sx} - 1 - sx)\Pi(dx)} \leq e^{cte^s}. \quad (22)$$

The property (21) allows to prove the following corollary from Lemma 8.

Corollary 9. (i) *The distribution of X_1 is long-tailed if and only if the distribution of $X_1^{(3)}$ is so. In both cases, $\mathbb{P}\{X_1 > x\} \sim \mathbb{P}\{X_1^{(3)} > x\}$ as $x \rightarrow \infty$.*

(ii) *The distribution of X_1^+ is strong subexponential if and only if the distribution $\frac{\Pi(dx)}{\Pi(1, \infty)}$ concentrated on $(1, \infty)$ is so. In both cases, $\mathbb{P}\{X_1 > x\} \sim \Pi(x, \infty)$ as $x \rightarrow \infty$.*

Proof. The assertion (i) is immediate from Lemma 8.

(ii) If X_1^+ has strong subexponential distribution, then it is particularly long-tailed, so that $\mathbb{P}\{X_1 > x\} \sim \mathbb{P}\{X_1^{(3)} > x\}$ as $x \rightarrow \infty$. Hence, the distribution of $X_1^{(3)+}$ is strong subexponential too, and in particular subexponential. Since $X_1^{(3)+}$

has compound Poisson distribution with parameter $\Pi(1, \infty)$ and jump distribution $\frac{\Pi(dx)}{\Pi(1, \infty)}$ concentrated on $(1, \infty)$, Theorem 3 of Foss et al. (2013) yields that $\mathbb{P}\{X_1 > x\} \sim \Pi(x, \infty)$ as $x \rightarrow \infty$. Therefore, the distribution $\frac{\Pi(dx)}{\Pi(1, \infty)}$ concentrated on $(1, \infty)$ is strong subexponential—see, e.g. Foss et al. (2013, Corollary 3.26).

If the distribution $\frac{\Pi(dx)}{\Pi(1, \infty)}$ concentrated on $(1, \infty)$ is strong subexponential, then $\mathbb{P}\{X_1 > x\} \sim \mathbb{P}\{X_1^{(3)+} > x\} \sim \Pi(x, \infty)$ by the theorem on tail behavior for random sums—see e.g. Foss et al. (2013, Theorem 3.37). \square

Proof of Theorem 6. We start with a lower bound. We have $a = \mathbb{E}X_1^{(3)} + \mathbb{E}Z_1$. Fix $\varepsilon > 0$ and consider two independent processes

$$X_t^\varepsilon := X_t^{(3)} + t\mathbb{E}Z_1 - t\varepsilon \quad \text{and} \quad Z_t^\varepsilon := Z_t - t\mathbb{E}Z_1 + t\varepsilon,$$

so that $X_t = X_t^\varepsilon + Z_t^\varepsilon$. Then

$$\max_{u \in [0, t]} X_u \geq \max_{u \in [0, t]} X_u^\varepsilon + \inf_{u \geq 0} Z_u^\varepsilon.$$

Therefore, for any x and $y > 0$,

$$\mathbb{P}\left\{\max_{u \in [0, t]} X_u > x\right\} \geq \mathbb{P}\left\{\max_{u \in [0, t]} X_u^\varepsilon > x + y\right\} \mathbb{P}\left\{\inf_{u \geq 0} Z_u^\varepsilon > -y\right\}.$$

The process Z_t^ε is positively driven, because $\mathbb{E}Z_t^\varepsilon = t\varepsilon > 0$. This yields that the overall minimum of the process Z_t^ε is finite with probability 1. In particular, there exists an $y_0 > 0$ such that

$$\mathbb{P}\left\{\inf_{u \geq 0} Z_u^\varepsilon > -y_0\right\} \geq 1 - \varepsilon,$$

which implies, for all $t > 0$,

$$\mathbb{P}\left\{\max_{u \in [0, t]} X_u > x\right\} \geq (1 - \varepsilon) \mathbb{P}\left\{\max_{u \in [0, t]} X_u^\varepsilon > x + y_0\right\}. \quad (23)$$

Since X_1^+ is assumed to be strong subexponential, by Corollary 9 the distribution $\frac{\Pi(dx)}{\Pi(1, \infty)}$ concentrated on $(1, \infty)$ is strong subexponential too and

$$\overline{\Pi}(x) \sim \mathbb{P}\{X_1 > x\} = \overline{F}(x) \quad \text{as } x \rightarrow \infty.$$

Then the compound Poisson process X_t^ε with drift $(a - \varepsilon)t$ satisfies all the conditions of Theorem 3 with τ 's exponentially distributed which implies

$$\mathbb{P}\left\{\max_{u \in [0, t]} X_u^\varepsilon > x\right\} \sim \frac{1}{|a - \varepsilon|} \int_x^{x+t|a-\varepsilon|} \overline{F}(v) dv$$

as $x \rightarrow \infty$ uniformly for all $t > 0$. Taking into account that

$$\int_x^{x+t|a-\varepsilon|} \overline{F}(v) dv \geq \int_x^{x+t|a|} \overline{F}(v) dv$$

and letting $\varepsilon \downarrow 0$, we conclude from (23) the lower bound

$$\mathbb{P}\left\{\max_{u \in [0, t]} X_u > x\right\} \geq \frac{1 + o(1)}{|a|} \int_x^{x+t|a|} \overline{F}(v) dv \quad \text{as } x \rightarrow \infty. \quad (24)$$

Now proceed to prove an upper bound. Consider two independent processes

$$X_t^\varepsilon := X_t^{(3)} + t\mathbb{E}Z_1 + t\varepsilon \quad \text{and} \quad Z_t^\varepsilon := Z_t - t\mathbb{E}Z_1 - t\varepsilon,$$

so that $X_t = X_t^\varepsilon + Z_t^\varepsilon$. Then

$$\max_{u \in [0, t]} X_u \leq \max_{u \in [0, t]} X_u^\varepsilon + \max_{u \in [0, t]} Z_u^\varepsilon. \quad (25)$$

Here the process Z_t^ε is negatively driven, $\mathbb{E}Z_t^\varepsilon = -t\varepsilon < 0$. This yields that the overall supremum of the process Z_t^ε is finite with probability 1. Since all positive exponential moments of Z_1^ε are finite, there exists a $\beta = \beta(\varepsilon) > 0$ such that $\mathbb{E}e^{\beta Z_1^\varepsilon} = 1$. Then, in particular, the Cramér estimate says that (see also Bertoin and Doney (1994))

$$\mathbb{P}\left\{\sup_{u \geq 0} Z_u^\varepsilon > x\right\} \leq e^{-\beta x}. \quad (26)$$

We also need more accurate upper bound for $\mathbb{P}\left\{\sup_{u \in [0, t]} Z_u^\varepsilon > x\right\}$ for small values of t . Notice that, for all $s > 0$, the process $e^{s(Z_t - \mathbb{E}Z_t)}$ is a positive submartingale, so Doob's inequality is applicable

$$\begin{aligned} \mathbb{P}\left\{\sup_{u \in [0, t]} Z_u^\varepsilon > x\right\} &\leq \mathbb{P}\left\{\sup_{u \in [0, t]} (Z_u - \mathbb{E}Z_u) > x\right\} \\ &\leq e^{-sx} \mathbb{E}e^{s(Z_t - \mathbb{E}Z_t)} \\ &= e^{-sx} e^{s^2 t \sigma^2 / 2} \mathbb{E}e^{sX_t^{(2)}}. \end{aligned}$$

Recalling the upper bound (22) for $\mathbb{E}e^{sX_t^{(2)}}$, we get

$$\mathbb{P}\left\{\sup_{u \in [0, t]} Z_u^\varepsilon > x\right\} \leq e^{-sx} e^{(s^2 \sigma^2 / 2 + ce^s)t}.$$

For $t \leq 1$, take $s := \log \frac{1}{t}$, then

$$\mathbb{P}\left\{\sup_{u \in [0, t]} Z_u^\varepsilon > x\right\} \leq c_1 e^{-sx} = c_1 t^x.$$

If $t \leq e^{-1}$, then we finally deduce

$$\mathbb{P}\left\{\sup_{u \in [0, t]} Z_u^\varepsilon > x\right\} \leq c_1 t t^{x-1} \leq c t e^{1-x} = c_2 t e^{-x}. \quad (27)$$

Since X_1 is assumed to be strong subexponential, by Corollary 9 the distribution $\frac{\Pi(dx)}{\Pi(1, \infty)}$ concentrated on $(1, \infty)$ is strong subexponential too. Then the compound

Poisson process X_t^ε with drift $(a + \varepsilon)t$ satisfies all the conditions of Theorem 3 with τ 's exponentially distributed and we have the following asymptotics

$$\begin{aligned} \mathbb{P}\left\{\max_{u \in [0, t]} X_u^\varepsilon > x\right\} &\sim \frac{1}{|a + \varepsilon|} \int_x^{x+t|a+\varepsilon|} \overline{F}(v) dv \\ &\leq \frac{1}{|a + \varepsilon|} \int_x^{x+t|a|} \overline{F}(v) dv \end{aligned} \quad (28)$$

as $x \rightarrow \infty$ uniformly for all $t > 0$. As follows from (26) and (27), uniformly for all $t > 0$,

$$\mathbb{P}\left\{\sup_{u \in [0, t]} Z_u^\varepsilon > x\right\} = o\left(\mathbb{P}\left\{\max_{u \in [0, t]} X_u^\varepsilon > x\right\}\right) \quad \text{as } x \rightarrow \infty. \quad (29)$$

Take any function $h(x) \rightarrow \infty$ such that $\overline{F}(x - h(x)) \sim \overline{F}(x)$ as $x \rightarrow \infty$ and consider the following upper bound

$$\begin{aligned} \mathbb{P}\left\{\max_{u \in [0, t]} X_u > x\right\} &\leq \mathbb{P}\left\{\max_{u \in [0, t]} X_u^\varepsilon > x - h(x)\right\} + \mathbb{P}\left\{\max_{u \in [0, t]} Z_u^\varepsilon > x - h(x)\right\} \\ &\quad + \mathbb{P}\left\{\max_{u \in [0, t]} X_u^\varepsilon + \sup_{u \in [0, t]} Z_u^\varepsilon > x, \ h(x) \leq \sup_{u \in [0, t]} Z_u^\varepsilon \leq x - h(x)\right\} \\ &:= P_1 + P_2 + P_3. \end{aligned} \quad (30)$$

Here the first probability P_1 on the right may be estimated as follows: by (28),

$$\begin{aligned} P_1 &\leq \frac{1 + o(1)}{|a + \varepsilon|} \int_0^{t|a|} \overline{F}(x - h(x) + v) dv \\ &\sim \frac{1}{|a + \varepsilon|} \int_0^{t|a|} \overline{F}(x + v) dv \end{aligned} \quad (31)$$

By (29) and (31),

$$P_2 = o\left(\mathbb{P}\left\{\max_{u \in [0, t]} X_u^\varepsilon > x - h(x)\right\}\right) = o\left(\int_0^{t|a|} \overline{F}(x + v) dv\right) \quad \text{as } x \rightarrow \infty. \quad (32)$$

The probability P_3 is not greater than

$$\begin{aligned} &\int_{h(x)}^{x-h(x)} \mathbb{P}\left\{\sup_{u \in [0, t]} X_u^\varepsilon > x - y\right\} \mathbb{P}\left\{\sup_{u \in [0, t]} Z_u^\varepsilon \in dy\right\} \\ &\leq \sum_{n=h(x)+1}^{x-h(x)} \mathbb{P}\left\{\sup_{u \in [0, t]} X_u^\varepsilon > x - n\right\} \mathbb{P}\left\{\sup_{u \in [0, t]} Z_u^\varepsilon \in [n-1, n]\right\} \\ &\leq c_1 \sum_{n=h(x)+1}^{x-h(x)} \mathbb{P}\left\{\sup_{u \in [0, t]} X_u^\varepsilon > x - n\right\} e^{-\beta n}, \end{aligned}$$

due to the exponential upper bound (26) for Z_u^ε . Then it follows from (28) that

$$\begin{aligned} P_3 &\leq c_2 \sum_{n=h(x)+1}^{x-h(x)} e^{-\beta n} \int_0^{t|a|} \overline{F}(x-n+v) dv \\ &\leq c_3 \int_0^{t|a|} dv \int_{h(x)}^{x-h(x)} \overline{F}(x+v-y) e^{-\beta y} dy. \end{aligned}$$

Since F is long-tailed, $e^{-\beta x} = o(\overline{F}(x))$. Together with $F \in \mathcal{S}^*$ this implies that

$$\int_{h(x)}^{x-h(x)} \overline{F}(x+v-y) e^{-\beta y} dy = o(\overline{F}(x+v)) \quad \text{as } x \rightarrow \infty,$$

so that

$$P_3 = o\left(\int_x^{x+t|a|} \overline{F}(v) dv\right) \quad \text{as } x \rightarrow \infty. \quad (33)$$

Substituting (31)–(33) into (30) we obtain that

$$\mathbb{P}\left\{\max_{u \in [0, t]} X_u > x\right\} \leq \frac{1 + o(1)}{|a + \varepsilon|} \int_x^{x+t|a|} \overline{F}(v) dv$$

as $x \rightarrow \infty$ uniformly for all $t > 0$. Letting $\varepsilon \downarrow 0$, we conclude the desired upper bound

$$\mathbb{P}\left\{\max_{u \in [0, t]} X_u > x\right\} \leq \frac{1 + o(1)}{|a|} \int_x^{x+t|a|} \overline{F}(v) dv \quad \text{as } x \rightarrow \infty. \quad (34)$$

Together with the lower bound (24) it implies the required asymptotics. \square

Similar to Theorem 5 we conclude with the following principle of a single big jump for the maximum M_t of the Lévy process X_t . Let T_k be the time epoch of the k th jump of the compound Poisson process $X_t^{(3)}$ with jump absolute values greater than 1 arising in the decomposition of X_t into three independent processes. Let λ be the intensity of this compound Poisson process and Y_k 's be its successive jumps. Let the events D_k be defined literally in the same way as in Theorem 5, see (13).

Theorem 10. *In conditions of Theorem 6, for any fixed $\varepsilon > 0$,*

$$\lim_{A \rightarrow \infty} \lim_{t, x \rightarrow \infty} \mathbb{P}\left\{\bigcup_{k=1}^{N_t} D_k \mid M_t > x\right\} \geq \frac{|a|}{|a| + 2\varepsilon/\lambda}.$$

4 Sampling of Lévy process

The last section result allows to derive tail asymptotics for a Lévy process X_t stopped at random time τ and for its maxima M_τ within this time interval.

Theorem 11. Assume that a positive random variable τ is independent of the Lévy process X_t . Let the distribution F of X_1 be strong subexponential. If $a := \mathbb{E}X_1 < 0$ then

$$\mathbb{P}\{M_\tau > x\} \sim \frac{1}{|a|} \mathbb{E} \int_x^{x+\tau|a|} \overline{F}(y) dy \quad \text{as } x \rightarrow \infty. \quad (35)$$

Assume in addition that $\mathbb{E}\tau < \infty$. Then

(i) If $\mathbb{E}X_1 < 0$ then

$$\mathbb{P}\{X_\tau > x\} \sim \mathbb{P}\{M_\tau > x\} \sim \mathbb{E}\tau \overline{F}(x) \quad \text{as } x \rightarrow \infty. \quad (36)$$

(ii) If $\mathbb{E}X_1 \geq 0$ and if there exists $c > \mathbb{E}X_1$ such that

$$\mathbb{P}\{c\tau > x\} = o(\overline{F}(x)) \quad \text{as } x \rightarrow \infty, \quad (37)$$

then asymptotics (36) again hold.

Proof. Conditioning on τ which is independent of X_t , we deduce that

$$\mathbb{P}\{M_\tau > x\} = \int_0^\infty \mathbb{P}\{M_t > x\} \mathbb{P}\{\tau \in dt\}.$$

Then by Theorem 6, as $x \rightarrow \infty$,

$$\mathbb{P}\{M_\tau > x\} \sim \frac{1}{|a|} \int_0^\infty \int_x^{x+t|a|} \overline{F}(v) dv \mathbb{P}\{\tau \in dt\}$$

and the first assertion (35) follows.

In our proof of (i) and (ii) we follow the proof of Theorem 1 in Denisov et al. (2010). Since $X_\tau \leq M_\tau$, it is sufficient to prove that

$$\liminf_{x \rightarrow \infty} \frac{\mathbb{P}\{X_\tau > x\}}{\overline{F}(x)} \geq \int_0^\infty t \mathbb{P}\{\tau \in dt\} = \mathbb{E}\tau \quad (38)$$

and

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}\{M_\tau > x\}}{\overline{F}(x)} \leq \mathbb{E}\tau. \quad (39)$$

Again conditioning on τ implies

$$\mathbb{P}\{X_\tau > x\} = \int_0^\infty \mathbb{P}\{X_t > x\} \mathbb{P}\{\tau \in dt\}.$$

By the subexponentiality of X_1 , here $\mathbb{P}\{X_t > x\}$ is equivalent to $t\overline{F}(x)$ as $x \rightarrow \infty$, regardless of the sign of $\mathbb{E}X_1$. Then (38) follows by Fatou's lemma.

Let us now prove (39). If $\mathbb{E}X_1 < 0$ then (39) follows from (35) by dominated convergence due to

$$\int_x^{x+|a|\tau} \overline{F}(v) dv \sim |a|\tau \overline{F}(x) \quad \text{as } x \rightarrow \infty$$

and upper bound

$$\int_x^{x+|a|\tau} \overline{F}(v)dv \leq |a|\tau \overline{F}(x).$$

In the case $\mathbb{E}X_1 \geq 0$, we start with the following upper bound: for any N ,

$$\begin{aligned} \mathbb{P}\{M_\tau > x\} &\leq \mathbb{P}\{M_\tau > x, \tau \leq N\} + \mathbb{P}\{M_\tau > x, \tau \in (N, x/c]\} + \mathbb{P}\{c\tau > x\} \\ &=: P_1 + P_2 + P_3. \end{aligned} \quad (40)$$

By Theorem 7, $\mathbb{P}\{M_t > x\} \sim \mathbb{P}\{X_t > x\} \sim t\overline{F}(x)$ as $x \rightarrow \infty$, for every t . In addition, $M_t \leq M_N$ for $t \leq N$. Thus, dominated convergence yields that, for any fixed N ,

$$\mathbb{P}\{M_\tau > x, \tau \leq N\} = \int_0^N \mathbb{P}\{M_t > x\} \mathbb{P}\{\tau \in dt\} \sim \mathbb{E}\{\tau; \tau \leq N\} \overline{F}(x) \quad \text{as } x \rightarrow \infty.$$

Therefore, there exists an increasing function $N(x) \rightarrow \infty$ such that

$$P_1 = \mathbb{P}\{M_\tau > x, \tau \leq N(x)\} \sim \mathbb{E}\tau \overline{F}(x). \quad (41)$$

In what follows, we consider the representation (40) with $N(x)$ in place of N . In order to estimate P_2 in (40) we take $\varepsilon = (c - \mathbb{E}X_1)/2 > 0$ and $b = (\mathbb{E}X_1 + c)/2$. Consider $\widetilde{X}_t := X_t - bt$ and $\widetilde{M}_t = \sup_{u \leq t} \widetilde{X}_u$. Then $\mathbb{E}\widetilde{X}_1 = -\varepsilon < 0$ and Theorem 6 is applicable. Taking into account that $M_t \leq \widetilde{M}_t + bt$, we obtain that there exists K such that, for all x and t ,

$$\begin{aligned} \mathbb{P}\{M_t > x\} &\leq \mathbb{P}\{\widetilde{M}_t > x - bt\} \\ &\leq K \int_0^{\varepsilon t} \widetilde{F}(x - bt + y) dy \\ &\leq K \int_0^{\varepsilon t} \overline{F}(x - bt + y) dy. \end{aligned}$$

Hence,

$$P_2 = \mathbb{P}\{M_\tau > x, \tau \in (N(x), x/c]\} \leq K \int_{N(x)}^{x/c} \mathbb{P}\{\tau \in dt\} \int_0^{\varepsilon t} \overline{F}(x - bt + y) dy.$$

Since $b - \varepsilon = \mathbb{E}X_1$,

$$\int_0^{\varepsilon t} \overline{F}(x - bt + y) dy = \int_{\mathbb{E}X_1 t}^{bt} \overline{F}(x - y) dy.$$

Then

$$\begin{aligned} P_2 &\leq K \int_{N(x)\mathbb{E}X_1}^{bx/c} \overline{F}(x - y) dy \int_{\max(N(x), y/b)}^{x/c} \mathbb{P}\{\tau \in dt\} \\ &\leq K \int_{N(x)\mathbb{E}X_1}^{bx/c} \overline{F}(x - y) \mathbb{P}\{\tau > y/b\} dy. \end{aligned} \quad (42)$$

Owing $b < c$ and the condition (37), the inequality $\mathbb{P}\{\tau > y/b\} \leq K_1 \bar{F}(y)$ holds for some K_1 and all y . Therefore,

$$P_2 \leq K K_1 \int_{N(x)\mathbb{E}\xi}^{bx/c} \bar{F}(x-y) \bar{F}(y) dy = o(\bar{F}(x)) \quad \text{as } x \rightarrow \infty \quad (43)$$

follows from $b/c < 1$ and from $F \in \mathcal{S}^*$, see, e.g. Foss et al. (2013, Theorem 3.24).

Finally, by the condition (37),

$$P_3 = \mathbb{P}\{c\tau > x\} = o(\bar{F}(x)) \quad \text{as } x \rightarrow \infty. \quad (44)$$

Substituting (41), (43), and (44) into (40) we conclude (39) and the proof is complete. \square

5 Application to ruin probabilities

The results obtained above are directly applicable to the *Cramér–Lundberg renewal model* in the collective theory of risk defined as follows (see e.g. Asmussen and Albrecher (2010, Sec. X.3)). We consider an insurance company and assume the constant inflow of premium occurs at rate c , that is, the premium income is assumed to be linear in time with rate c . Also assume that the claims incurred by the insurance company arrive according to a renewal process N_t with intensity λ and the sizes (amounts) $Y_n \geq 0$ of the claims are independent identically distributed random variables with common distribution B and mean b . The Y 's are assumed to be independent of the process N_t . The company has an initial risk reserve $u = R_0 \geq 0$.

Then the risk reserve R_t at time t is equal to

$$R_t = u + ct - \sum_{i=1}^{N_t} Y_i.$$

Then the probability

$$\begin{aligned} \psi(u, t) &:= \mathbb{P}\{R_s < 0 \text{ for some } s \in [0, t]\} \\ &= \mathbb{P}\left\{\min_{s \in [0, t]} R_s < 0\right\} \end{aligned}$$

is the finite time horizon probability of ruin. The techniques developed for compound renewal process with drift provide a method for estimating the probability of ruin in the presence of heavy-tailed distribution for claim sizes. We have

$$\psi(u, t) = \mathbb{P}\left\{\sum_{i=1}^{N_s} Y_i - cs > u \text{ for some } s \in [0, t]\right\}.$$

Since $c > 0$, the ruin can only occur at a claim epoch. Therefore,

$$\psi(u, t) = \mathbb{P}\left\{\sum_{i=1}^n Y_i - cT_n > u \text{ for some } n \leq N_t\right\},$$

where T_n is the n th claim epoch, so that $T_n = \tau_1 + \dots + \tau_n$ where the τ 's are independent identically distributed random variables with expectation $1/\lambda$. The last relation represents the ruin probability problem as the tail probability problem for the maximum of a compound renewal process with drift.

Let the *net-profit condition* $c > b\lambda$ hold, thus the process has a negative drift and $\psi(u, t) \rightarrow 0$ as $u \rightarrow \infty$, uniformly for all $t \geq 0$. Applying Theorem 3(i), we deduce the following result on the decreasing rate of the ruin probability to zero as the initial risk reserve becomes large in the case of heavy-tailed claim size distribution, compare with a result for fixed t in Section X.4 in Asmussen and Albrecher (2010) and with Theorem 5.21 for the compound Poisson model in Foss et al. (2013).

Theorem 12. *In the compound renewal risk model, let $c > b\lambda$. If the claim size distribution B is strong subexponential, then, uniformly for all $t \geq 0$,*

$$\psi(u, t) \sim \frac{\lambda}{c - b\lambda} \int_u^{u+t(c/\lambda-b)\mathbb{E}N_t} \overline{B}(v) dv \quad \text{as } u \rightarrow \infty.$$

In particular,

$$\psi(u, t) \sim \frac{\lambda}{c - b\lambda} \int_u^{u+t(c-b\lambda)} \overline{B}(v) dv \quad \text{as } u, t \rightarrow \infty.$$

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